Wave Equations:

see Flygare: Chap 1

Maxwell eqn: (SI)

$(\varepsilon_0 \mu_0)^{-1} = c^2$

1. $\vec{\nabla} \times \vec{E} + \partial \vec{B}/\partial t = 0$ $\qquad \vec{B} = \mu_0 \mu_p \vec{H}$ $\qquad$ Vacuum $\mu_p = \varepsilon = 1$
2. $\vec{\nabla} \times \vec{H} - \partial \vec{D}/\partial t = \vec{J}$ $\qquad \vec{D} = \varepsilon_0 \varepsilon \vec{E}$ $\qquad \mu_0, \varepsilon_0$ const
3. $\vec{\nabla} \cdot \vec{D} = \rho$ $\qquad \vec{J} = \rho \vec{v}$ current density
4. $\vec{\nabla} \cdot \vec{B} = 0$ $\qquad$ recall: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ grad

(no magnetic monopoles)

$\vec{\nabla}\cdot$ divergence, $\vec{\nabla}\times$ curl

$\nabla \cdot \nabla = \nabla^2$ LaPlacian operator

1 → (Faraday) time dep $\vec{B}$ → induce $\vec{E} \perp \vec{B}$ (since $\nabla \times E$ cancels $\dot{B}$)

2 → (Ampere - Oersted) ~~time dependent~~ electric (displacement) field produces $\vec{B}$ and $\vec{B}$ exist near a current

3 → (Coulomb law) field relate to charge

4 → no magnetic monopoles

Define scalar + vector potentials

ϕ: static (time independent) $\quad$ 1. $\vec{\nabla} \times \vec{E} = 0 \longrightarrow \vec{E} = -\vec{\nabla}\phi$

fits eqn 1: $(\vec{\nabla} \times (\vec{\nabla}\phi) = 0$ - due to cross prod)

note: $\nabla \cdot D = \nabla \cdot (\varepsilon_0 \varepsilon E) = \nabla \cdot \nabla(\varepsilon\varepsilon_0 \phi) \Rightarrow \nabla^2 \phi = \frac{1}{\varepsilon\varepsilon_0}\rho$

(vacuum: LaPlace: $\nabla^2\phi = 0$) $\qquad$ Poisson's eqn

$\vec{A}$: time varying consideration $\quad$ 2. $E = -\frac{\partial A}{\partial t} - \nabla\phi$

into #1 $\quad -\nabla \times \left(\frac{\partial A}{\partial t} + \nabla\phi\right) + \frac{\partial B}{\partial t} = 0$

$\frac{\partial}{\partial t}(\nabla \times A) = \frac{\partial}{\partial t} B \Rightarrow B = \nabla \times A$

Constrain variables

Scalar ↔ vector: (5) $\nabla \cdot A + \varepsilon\mu_p \varepsilon_0 \mu_0 \frac{\partial\phi}{\partial t} = 0$ (Lorentz convention) $\left\{ \begin{array}{l} \phi \neq \phi(t) \\ \nabla \cdot A = 0 \end{array} \right.$

Use these with Maxwell (3) $\quad \vec{\nabla} \cdot (\varepsilon_0 \varepsilon \vec{E}) = \rho = \vec{\nabla} \cdot \left(-\frac{\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right) \varepsilon_0 \varepsilon$

$-\rho/\varepsilon\varepsilon_0 = \left[\frac{\partial}{\partial t}(\nabla \cdot A) + \nabla^2\phi\right]$

from above $\frac{\partial}{\partial t}$(5): $\frac{\partial}{\partial t}(\nabla \cdot A) + \varepsilon\mu_p\varepsilon_0\mu_0 \frac{\partial^2}{\partial t^2}\phi = 0$

plug in + rearrange: $\boxed{-\nabla^2\phi + \frac{\varepsilon\mu}{c^2}\frac{\partial^2\phi}{\partial t^2} = \frac{\rho}{\varepsilon\varepsilon_0}}$

wave equation for the scalar potential

Next take eqn(2): $\vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t} = \vec{J}$

substitute: $\frac{1}{\mu\mu_0} \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \varepsilon\varepsilon_0 \frac{\partial}{\partial t}\left(\frac{-\partial \vec{A}}{\partial t} - \vec{\nabla}\phi\right) = \vec{J}$

$$(\nabla \times \nabla \times A) + \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2} A + \frac{\partial}{\partial t}\nabla\phi = \mu\mu_0 \vec{J}$$

$$\nabla(\nabla \cdot A) - \nabla^2 A \qquad \nabla \frac{\partial \phi}{\partial t}$$

$$-\nabla^2 A + \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2} A + \nabla\left(\nabla \cdot A \frac{-1}{\varepsilon\varepsilon_0\mu\mu_0} \frac{\partial \phi}{\partial t}\right) = \mu\mu_0 \vec{J}$$

$$\boxed{\nabla^2 A - \frac{\mu\varepsilon}{c^2}\frac{\partial^2}{\partial t^2} A = -\mu\mu_0 \vec{J}}$$

If $J = 0$ then $\nabla^2 A = \frac{\varepsilon\mu}{c^2}\frac{\partial^2}{\partial t^2} A$

let $A = A_0 e^{-i[k \cdot r - \omega t]}$

$$\nabla^2 A = \nabla\left(-ik e^{-i(k \cdot r - \omega t)}\right) = -k^2 e^{-i[k \cdot r - \omega t]} = -k^2 A$$

$$\frac{\partial^2}{\partial t^2} A = \frac{\partial}{\partial t}\left(+i\omega e^{-i(k \cdot r - \omega t)}\right) = -\omega^2 e^{-i[k \cdot r - \omega t]} = -\omega^2 A$$

$$-k^2 A = -\frac{\varepsilon\mu}{c^2}\omega^2 A$$

$$\underline{k} = (\varepsilon\mu)^{1/2}\left(\frac{\omega}{c}\right) = (\varepsilon\mu)^{1/2}\left(\frac{2\pi\nu}{c}\right) = (\varepsilon\mu)^{1/2}\left(\frac{2\pi}{\lambda}\right)$$

note: $\mu_p = 1$ (non magnetic medium) $\varepsilon^{1/2} = n$ refractive index

propagation vector $\boxed{|\vec{k}| = 2\pi n/\lambda}$ speed of light c/n

Poyting vector $\vec{S} = \vec{E} \times \vec{H}$ in vacuum $\underline{n = 1.0}$

Now use equations

$$B = \nabla \times A = \nabla \times A_0 e^{i(k \cdot r - \omega t)} = -A_0 \times \nabla e^{i[k \cdot r - \omega t]}$$

$$\vec{B} = -i \vec{A} \times \vec{k} e^{i[k \cdot r - \omega t]} \qquad B \perp A$$

$$\boxed{B = B_0 e^{i(k \cdot r - \omega t)}}$$

$$E = -\frac{\partial}{\partial t} A = -\omega\{i A_0 e^{+i(k \cdot r - \omega t)}\}$$

$$\boxed{\vec{E} = \vec{E}_0 e^{i(k \cdot r - \omega t)}} \qquad E \parallel A$$

$\vec{E}, \vec{B}$ time oscillating (ω) & spatial varying (k) fields $\Rightarrow$ $\boxed{\text{E-M radiation}}$

~~Vacuum: $J = 0$, $\rho = 0$ $\nabla \cdot (\nabla \times E) + \nabla \cdot \frac{\partial B}{\partial t} = 0$~~