Angular Momentum – Levine Ch 5 – all the arithmetic for rotational syst

We have worked out several problems whose energy eight functions were also eigen functions of the angular momentum operators:

$L_z e^{im\varphi} = mhe^{im\varphi}$

$L_z Y_{\_m} = mh\, Y_{\_m}$

$L^2 Y_{\_m} = \_(\_+1)h^2\, Y_{\_m}$

Angular momentum is a natural conserved quantity of “spherical” systems _ central force ( ) (as rotation of rigid body, electro static attire) just as linear momentum natural property of linear syst (e.g. Newton’s laws)

We have seen that $[L^2, L_z] = 0$ because they have a set of simultaneous eigen functions and also noted $[L_x, L_y] = ihL_z$ (etc., x,y,z rotate) where $\vec{L}$ is a vector operator:

$\vec{L} = \hat{L}_x\vec{i} + \hat{L}_y\vec{j} + \hat{L}_z\vec{k}$ $\qquad \vec{i}, \vec{j}, \vec{k}$ unit vector

note parallel:

$\vec{\ } = \frac{f}{fx}\vec{i} + \frac{f}{fy}\vec{j} + \frac{f}{fz}\vec{k}$ $\qquad$ gradient operator

vector properties:

data on scalar prod: $\vec{L}\,?\vec{L} = L_x^2 + L_y^2 + L_z^2$ $\qquad \vec{A}\,?\vec{B}\ {}_z a_x b_x + a_y b_y + a_z b_z$

$\vec{L} \leftrightarrow \vec{L} = 0, \quad \vec{L}_1 \leftrightarrow \vec{L}_2 = \vec{i}(L_{y_1}L_{z_2} - L_{z_1}L_{y_2}) +$

$\vec{j}(L_{z_1}L_{x_2} - L_{x_1}L_{z_2}) + \vec{k}(L_{x_1}L_{y_2} - L_{y_1}L_{x_2})$

Classically vector on cross prod:

$\vec{L} = \vec{r} \leftrightarrow \vec{p} = \vec{i}(yp_z - zp_y) + \vec{j}(zp_x - xp_z) + \vec{k}(xp_y - yp_z)$

converting to q.m.

$L_x = -i\hbar\,{}^{3\!/\!4}\,y\frac{f}{fz} - z\frac{f}{fy}\surd, \quad L_y = -i\hbar\left(z\frac{f}{fx} - x\frac{f}{fz}\right), \quad L_z = -i\hbar\,{}^{3\!/\!4}\,x\frac{f}{fy} - y\frac{f}{fx}\surd$

Plugging these into eqn for L^2 can demonstrate:

$[L^2, L_x] = 0 = [L^2, L_y]$ see Levine p 89

so why don’t we specialize on L_x? We can

totally arbitrary to use L_z, but arith convert it

but cannot do L_x and L_z etc because $[L_x, L_z] = -ihL_y$ etc (note order)

In (r,θ,φ) corrdinates this is cleaner _ by definition choose θ to be rotation about z (arb) but now one axis differs from other 2:

$$\hat{L}^2 = -\hbar^2\hat{\Lambda}^2 = -\hbar^2 ¾ \frac{1}{\sin^2\theta}\frac{f}{f\varphi^2} + \frac{1}{\sin\theta}\frac{f}{f\theta}\sin\theta\frac{f}{f\theta}√$$

see that onlyl z φ rotation is singled out and

$$\left[\hat{L}^2, \hat{L}_z\right] = -\hbar^2\Lambda^2, -i\hbar\frac{f}{f\varphi} = 0 \quad \text{since} \quad \frac{f^2}{f\varphi^2}, \frac{f}{f\varphi} = 0 \text{ and } f(\theta), \frac{f}{f\varphi} = 0$$

Levine 5.4

So what we have here are a number of operators with very well-defined relationships _ turns out that alone sufficient to define ang mom and do not need form of operator to understand:

Assume here vector operator $\hat{M} = \hat{M}_x\vec{i} + \hat{M}_y\vec{j} + \hat{M}_z\vec{k}$ with these properties:

$$\left[\hat{M}_x, \hat{M}_y\right] = i\hbar\hat{M}_z, \quad \left[\hat{M}_y, \hat{M}_z\right] = i\hbar\hat{M}_z, \quad \left[\hat{M}_z, \hat{M}_x\right] = i\hbar\hat{M}_y$$

now can write: $\hat{M}^2 = \hat{M} ? \hat{M} = \hat{M}_x{}^2 + \hat{M}_y{}^2 + \hat{M}_z{}^2$

solve: $\left[\hat{M}^2, M_z\right] = \left[M_x{}^2, M_z\right] + \left[M_y{}^2, M_z\right] + \left[M_z{}^2, M_z\right]$

= $M_xM_xM_z – M_zM_xM_x + M_yM_yM_z – M_zM_yM_y$

= $M_x[M_x, M_z] + [M_x, M_z]M_x + M_y[M_y, M_z] + [M_y, M_z]M_y$

$$\left[M^2, \hat{M}_z\right] = (-i\hbar M_y)M_x + (-i\hbar M_y)M_x + (i\hbar M_x)M_y + i\hbar M_xM_y = 0$$

See that can do same bit for: $[M^2, M_x] = [M^2, M_y] = 0$

so knew that M^2 and M_x or M_y or M_z

will have simultaneous eigen functions

Levine calls these Y: $\hat{M}^2 Y = aY$
$\hat{M}_2 Y = bY$

Now define a new operator $M_+ = M_x + iM_y$

$M_- = M_x - iM_y$

Investigate: $M_xM_- = (M_x + iM_y)(M_x – iM_y)$

= $M_x{}^2 + M_y{}^2 + i[M_y, M_x]$

= $M^2 – M_z{}^2 + i[M_y, M_x](-ihM_y)$

$= M^2 - M_z^2 + ihM_z$

same method: $M_-M_+ = M^2 - M_z^2 + hM_z$

Also: $[M_+, M_-] = M_+M_- - M_-M_+ = 2hM_z$

$[M_+, M_z] = [M_x, M_z] + i[M_y, M_z]$

$= -ihM_y - hM_x$

$= -hM_+$

or $M_+M_z = M_zM_+ - hM_+$

similarly $M\text{-}M_z = M_zM_- + hM_-$

Now operate both sides on Y: $M_+M_zY = (M_zM_+ - hM_+)Y$

$bM_+Y = M_+(bY) = M_z(M_+Y) - h(M_+Y) = M_z - h)(M_+Y)$

rearrange: $M_z(M_+Y) = (b + h)(M_+Y)$

so (M_+Y) is eigen fct of M_z with eigen value $(b + h)$

Raising a Lowery operator effect:

$M_z(M_+Y) = (b + h)M_+Y$ or $M_+|jk\rangle = c^+_{jk}\hbar|p,k+1\rangle$

if repeat: $M_z(M_+^2Y) = (b + 2h)(M_+^2Y)$ or $M-|pk\rangle = c^-_{jk}\hbar|p,k-1\rangle$

$M_z(M_-^n Y) = (b - nh)(M_-^n Y)$

Now we have a set of eigen values and eigen functions of M_z that are all h apart:
b-2h, b-h, b, b+h, . . . and since $[M^2, M_z] = 0$ must also be eigen fct M^2 but can show all same eigen value of M^2

$M^2[M_+^n Y] = a[M_+^n Y]$ n = 0,1,2, . . .

Know: $[M^2, M_+] = 0 = [M^2, M_x] + I[M^2, M_y] = 0 + 0$

can also show $[M^2, M_+^n] = 0$

Now: prones all same L^2 eigen value a: $M^2\left(M_\pm^n Y\right) = M_\pm^n\left(M^2Y\right) = M_\pm^n aY = a\left(M_\pm^n Y\right)$

or $M^2M_\pm^n|p,k\rangle = a_{jk+n}c^{\pm}_{jkn}\hbar^2|p,k\pm n\rangle$

So here sets of eigen fct and set eigen values but so far infinite. Must be same sine same eigen value

call then $Y_n^{\pm} = M_{\pm}^n Y$ $\quad M_z Y_n^{\pm} = b_n^{\pm} Y_n^{\pm}$ $\quad b_n^{\pm} = b_l n\hbar$

Bounds from $M^2 = M_x^2 + M_y^2 + M_z^2$ (intrinsically positive quant)

$M^2 – M_z^2 = M_x^2 + M_y^2$

operate on Y_n: $(M^2 – M_z^2)Yn = (M_x^2 + M_z^2)Y_n$

$(a_p – h_k^2)h^2/n \geq 0$

$b_K^2 \leq a$ ♦ $b_K \leq \sqrt{a_p^1}$ or $a_p^{n/2} \leq b_K \leq a_p^{1/2}$

Now know that $|b_K|$ limited $< \sqrt{a_p}$ but what is it

one of these is max b_{max} ? $M_+|p,k_{max}\rangle = 0$

so similarly $M^-M^+|p,k_{max}\rangle = 0$

M^-M^+ established as: $\left(M^2 - M^2 - \hbar M_z\right)|p,k_{max}\rangle = 0$

$M^2|p,k_{max}\rangle = \left(M^2 + \hbar M_z\right)|p,k_{max}\rangle$

therefore $\quad a_{pk}\hbar^2 = \left(b_{K\,max}^2 + b_K^2\right)\hbar^2 =$

$a_{jk} = b_{K\,max}\left(b_{K\,max}^2 + 1\right)$

Know that operate M- on $|pk_{max}\rangle$

get new fct ~$|p,k_{max} - 1\rangle$

repeat n times $|p,k_{max} - n\rangle$ in steps of 1

so looks like k_{max} special call it “j”

b_K = j, j-1, j-2, . . . $\qquad$ and $a_{pK} = j(j + 1)$

Also a <u>minimum</u> for $|b_K| < a$

same idea: $M^+M^-|p,k_{min}\rangle = 0$

$\left(M^2 - M_z^2 - \hbar M^2\right)|p,k_{min}\rangle = 0$

$M^2|p,k_M\rangle = b_{K_{min}}\left(b_{K_M} - 1\right)\hbar^2 = a_{pK}\hbar^2 =$

or $b_{K_{min}}(b_{K_{min}} – 1) = j(j + 1)$

solu $b_{K_{min}} = -j$

Now see something special <u>ladder operators</u> shift

$M_-|j,k_{max}\rangle = c_{jK}|j\ k_m - 1\rangle \cdots$ in steps of k = 1

but minimum and max are same magnitude

so b_K = j, j-1, j-2, . . ., -j+1, -j = M an integer or half integer

means M has possibilities of being $M = 0,\ \pm\frac{1}{2},\ \pm 1,\ \pm\frac{3}{2}\ \ldots$ _ integer (ang mom), half integer (sum)

So now: $M^2|j\ m\rangle = j(j+1)\hbar^2|j\ m\rangle$

$M_2|j\ m\rangle = m\hbar^2|j\ m\rangle$

But what is result of M?

$M_-M_+|j\ m\rangle = \left(M^2 - M_z^2 - \hbar M_z\right)jm\rangle$

$c^-_{j,m+1}\ c^+_{j,m}\hbar^2|jm\rangle = \left[j(j+1) - m(m+1)\right]\hbar^2|jm\rangle$

Know product but not each one so:

$\langle jm|M_-|j\ m+1\rangle = c^-_{j,m+1}\hbar\langle jm|jm\rangle = c^-_{j,m+1}\hbar$

$= \langle jm|M_x|j\ m+1\rangle - i\langle jm|M_y|j\ m+1\rangle$

$= \langle j_{m+1}|M_x|jm\rangle^* - i\langle j_{m+1}|M_y|jm\rangle^*$

$= \left\{\langle j,m+1|M_x|jm\rangle + i\langle j,m+1|M_y|jm\rangle\right\}^*$ M_x, M_y hermitian: (not M_+, M_- not observable)

$= \langle jm+1|M_+|jm\rangle^* = c^{+\ *}_{jm}$

so $(c_{jm}^+)^* = (c_{j\ m+1}^-)$ since $\langle a|M_+|b\rangle = \langle b|M_-|a\rangle^*$ call: <u>hermitian conjugate</u>

$c_{j\ m+1}^-\ c_{j\ m}^+$ = j(j+1) – m(m+1)

$(c_{jm}^+)^*\ c_{j\ m}^+$ = [j(j+1) – m(m+1)]

$(c_{jm}^+)\ c_{j\ m}^+$ = [j(j+1) – m(m+1)]$^{1/2}$

similarly (c_{jm}^-) = [j(j+1) – m(m-1)]$^{1/2}$

result: $M_\pm|jm\rangle = \left[j(j+1) - m(m\pm 1)\right]^{1/2}\hbar|jm\rangle$

Note: try now: $M_+|jm_{max}\rangle = \left[j(j+1) - m_m(m_n \pm 1)\right]^{1/2}\hbar|j\ m+1\rangle = 0$

_ $m_{max} = j$

similarly $M_-|jm_{min}\rangle = \left[j(j+1) - m_m(m_n \pm 1)\right]\hbar = 0$? $m_{min} = -j$

Form of the eigen fct – all above was abstract and totally general but if we want the solution (fct form) of $|jm\rangle$ we need to get rep for M_+, M_-, M_z

but having those only med $|j,j\rangle$ all rest available by $M^k|j,j\rangle$ = (const)$|j\ j-k\rangle$

(const) = $(c^-_{j\,m}\ c^-_{j\,m-1} \cdots c^-_{j\,m-(k-1)})$

try: $M_z = -i\hbar \frac{f}{f\varphi}$

$M_x = -i\hbar\ \sin\varphi \frac{f}{f\varphi} + \cot\theta\ \cos\varphi \frac{f}{f\varphi}$?

$M_y = -i\hbar\ \cos\varphi \frac{f}{f\varphi} - \cot\theta\ \sin\varphi \frac{f}{f\varphi}$?

$M_+ = \hbar e^{i\varphi}\ \frac{f}{f\varphi} + i\cot\theta \frac{f}{f\varphi}$?

$M_- = -\hbar e^{-i\varphi}\ \frac{f}{f\varphi} - i\cot\theta \frac{f}{f\varphi}$?

since $M_+|jj\rangle = 0 = \hbar e^{i\varphi}\ \frac{f}{f\varphi} + i\cot\theta \frac{f}{f\varphi}?\Psi_{jj}(\theta,\varphi)$

so subst $\Psi(\theta,\varphi) = \Theta(\theta)\ \Phi(\varphi)$

can show $\frac{\tan\theta}{\Theta}\frac{d\Theta}{d\theta} = -i\frac{1}{\Phi}\frac{d\Phi}{d\varphi}$ separates so each = m (1st order diff)

$\tan\theta \frac{d\Theta}{d\theta} = m\Theta$ $\frac{d\Theta}{d\theta} = im\Phi$

$\Theta \sim \sin^m\theta$ $\Phi = e^{im\varphi}$ solve $M_z\Psi_{jj} = jh\Psi_{jj}$

$\Psi_{jj}(\theta,\varphi) = N \sin^m\theta\ e^{im\varphi}$ here m = j

all the rest: Y_{jm} by operate M-Ψ_{jj} successively

Spin _ Uhlenbeck & Goudsmit realized that if electron had intrinsic angular momentum and assoc. magnetic moment with 2 states = j = _ m = _, -_

so $S_z|\frac{1}{2},\pm\frac{1}{2}\rangle$? $\pm\frac{1}{2}\hbar|\frac{1}{2},\pm\frac{1}{2}\rangle$

Then could explain atomic spectra and Zeeman perturbation of spectra

Half-integer spin (intrinsic magnetic moment) came about naturally from the Dinae equation which accounted for relativistic effects

Electron "spin" so important define: $\alpha = \left| \tfrac{1}{2}\ \tfrac{1}{2} \right\rangle$

$S_z\alpha = \_h\alpha$ $S^2\alpha = \_h^2\gamma$ $\beta = \left| \tfrac{1}{2}\ -\tfrac{1}{2} \right\rangle$

$S_z\beta = -\_h\beta$ $S^2\beta = \_h^2\beta$

and $S_+\alpha = 0$ $S_-\alpha = h\beta$ $S_+\beta = h\alpha$ $S^-\alpha = 0$

See to connect α, β: $\langle \alpha | S^+ | \beta \rangle = \hbar$, $\langle \beta | S^- | \alpha \rangle = \hbar$

Note – no functional form needed yet now all we need to describe angular momentum and Zeeman perturbation

energy of field B^+ magnetic dipole μ_m

$E = -\vec{\mu}_m ?\vec{B}$? $H = -\vec{\mu}_m ?\vec{B} = +g\, {}^{\beta}\!/\!{}_{\hbar}\, \vec{S} ?\vec{B}$ $g_e \sim 2.0023$

$\mu_s = g_e\beta$ $\beta_e = 1h/2m_e = 9.27 \times 10^{-24}$ J/T

if $B = B_2$ (i.e., unique axis) $H_{B_2} = +g\, {}^{\beta}\!/\!{}_{\hbar}\, B ?\hat{M}_z$

Bohr magnetor mag of mag mom for elect with ang mom

$\mu_L = \beta_e(L(L+1))^{-}$

Levine Ch 11, 4, . . .; Atkins Ch 6, 6-7

More than one particle with angular momentum:

if 2 particles j,2 with ang mom $\vec{j}_1, \vec{j}_2$

what can we know about them (simultaneously)

since $[j_{1x}, j_{2x}] = 0$ and same for all x,y,z

since $\vec{j}_1, \vec{j}_2$ depend on diff coord syst

similarly $[j_1^2, j_2^2] = 0$ and all combine with x,y,z

so state: $\left| j_1, m_1; j_2, m_2 \right\rangle$ should satisfy syst, i.e. each fully specified

This should tell us the quantum numbers of state for each particle independently

? can we discuss total angular momentum?

$\vec{j} = \vec{j}_1 + \vec{j}_2$ and $\vec{j}_x = \vec{j}_{1x} + \vec{j}_{2x}$, etc.

natural way to write but is it an ang. mom?

yes since: $[j_x, j_y] = [j_{1x} + j_{2x}, j_{1y} + j_{2y}]$

$= [j_{1x}, j_{2x}] + [j_{1y}, j_{2y}]$ + zeros

$= ih(j_{1z} + j_{2z}) = ihj_z$

(neat aspect of general prop:)

so know: $[j^2, j_z] = 0$

total ang mom: $\sqrt{j(j+1)\hbar^2}$

j=0, _, 1, 3/2, . . .

$m_j = -j, -j+1, \ldots, j$

but what are j, m if know $j_{1,}, j_{2,}, m_{1,}, m_2$?

Show: $[j_2, j_1^2]=[j_2, j_2^2] = 0$ so j_2+j_2, j_2^2 simult.

$j^2|j_1m_1, j_2m_2\rangle = j(j+1)\hbar$

and $[j^2,j_z]=[j^2,j_z]=[j_z^2,j_z] = 0$ simult.

So know that can specify total ang mom, component totals, and projection on z of tot on

$j^2|j_1j_2jm\rangle = j(j+1)\psi^2|j_1j_2jm\rangle$

$j_2|j_1j_2jm\rangle = m\hbar|j_1j_2jm\rangle$

But cannot specify m_1, m_2, with all the above because

$[j^2, j_{1z}] = 2ih(j_{1y}j_{2x} - j_{1x}j_{2y}) \neq 0$

So can choose: $|j_1j_2jm\rangle$ or $|j_1m_1, j_2m_2\rangle$

as rep of 2 coupled angular momentor

now must know how to go back and forth and know what are values for j,m?

$j_2|j_1m_1, j_2m_2\rangle = (j_{z_1} + j_{z_2})|j_1m_1, j_2m_2\rangle$

$m\hbar|j_1m_1, j_2m_2\rangle = (m_1 + m_2)\hbar|j_1m_1, j_2m_2\rangle$

so: $m = m_1 + m_2$

makes sense, projection of each on z add up

now each m_1, m_2 have <u>maxima</u>: j_1, j_2
So maximum $m = j_1+j_2$
But this must be $j = j_1+j_2$ (one allowed value)
i.e. expect j must not be bigger than j_1+j_2
but incomplete since (2j+1) states $|j_1 m_1;\ m\rangle$? $(2j_1 + 2j_2 + 1)$
but $(2j_1+1)(2j_2+1)$ states $|j_1 m_1;\ j_2 m_2\rangle$? $2j_1 + 2j_2 + 4j_1 j_2$
so 4j, j_2 states need to be found, must have different j-value

Test: Max 75/most in 50-35 range   A – 60, B – 40, C
Memorize proofs – ok; need work on physical concepts, lots of errors on #3.
Problem is during appropriate work to answer question -- matter of experience

Review- coupling independent angular momentum

Established that $[M_x, M_y]=ihM_z$ for $\vec{M} = \hat{M}_{\bar{x}}\bar{i} + \hat{M}_{\bar{y}}\bar{j} + \hat{M}_{\bar{z}}\bar{k}$
Makes an ang mom $M_2|j\,m\rangle = m\hbar|j\,m\rangle$
$M^2|j\,m\rangle = j(j+1)\hbar^2|j\,m\rangle$
$M_\pm|j\,m\rangle = \{j(j+1)\ m(m\pm 1)\}^{1/2}|j\,m\pm 1\rangle$ m = j, j-1, . . ., -j
all done without recourse to any functional form $|j\,m\rangle$

<u>Levine Ch 11</u>
With multiple particles can have coupling of angular momentum--total is variable in question

since independent coord all op commute so state $|j_1 m_1, j_2 m_2\rangle$
$\vec{j} = \vec{j}_1 + \vec{j}_2$ is ang mom since $[j_x, j_y] = ihj_2 = ih(j_{21} + j_{22})$
show also: $[j^2, j_2] = 0$ and m = -j, -j+1, . . ., j; j = 0, 1/2, 1, 3/2, . . .
$[j^2, j_1^2] = [j^2, j_2^2] = 0$ $|j_1 j_2\, j\, m\rangle$ <u>should rep step</u>
$j^2|j_1 m j_2\, j\, m\rangle = j(j+1)\hbar^2|j_1 j_2\, j\, m\rangle$

$j_2 \left| j\, m j_2\, j\, m \right\rangle = m\hbar \left| j_1 j_2\, j\, m \right\rangle$

choose between: $\left| j_1\, m_1, j_2\, m_2 \right\rangle$ and $\left| j_1\, j_2\; j\, m \right\rangle$

as representations – diff situation each more ???

since $j_2 \left| j_1\, m_1\, j_1\, m_1 \right\rangle = (j_{12} + j_{22}) \left| j_1\, m_1, j_2\, m_2 \right\rangle = (m_1 + m_2)\hbar \left| j_1\, m_1, j_2\, m_2 \right\rangle$

$= m\hbar \left| j_1\, m_1, j_2\, m_2 \right\rangle$

Then $m = m_1 + m_2$ _ projection on z sum

$m_1(\max) = j_1$, $m_2(\max) = j_2$ _ $m(\max) = j_1 + j_2$ _ $j = j_1 + j_2$

account for degeneracy _ (2j + 1) (2j + 1)(2j₂ + 1) =

$(2j_1 + 2j_2 + 1) < 4j_1j_2 + (2j_1 + 2j_2 + 1)$

so need more than one j value

Next Ch. 8, 9 Approx methods Part Theory (9), Variation (8)

Consider m again: max m $m_m = m_1 + m_2$

next one down: $\left| j,\, m_m - 1 \right\rangle = \left| j_1 (m_1 - 1),\; j\, m_2 \right\rangle$ or $\left| j_1\, m_1,\; j_2 (m_2 - 1) \right\rangle$

so 2 possibilities, one is $\left| j,\, m_m - 1 \right\rangle$

and other is $\left| j',\, m_m - 1 \right\rangle$

now no other higher m than m-1 for j'

so j = m – 1

continue and get: $j = j_1 + j_2,\ j_1 + j_2 - 1,\ \ldots |j_1 - j_2$

limited since j must be positive

ex: $j_1 = 1$, $j_2 = 1$ j = 2,1,0

(2·1+1)(2·1+1) = 9 states 5+3+1 = 9 states

or $j_1 = 1$, $j_2 = 1/2$ for p-elect

$3 \cdot 2 =$ 6 states $j = 3/2, 1/2$ $4+2 =$ 6 states

Sometimes called the triangle condition (vector addition)
sides must match up with j_1, j_2, j all integers

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leads to the vector model of angular momentum particularly used a lot for spin
a) length $\sqrt{j(j+1)}$, same for components: $\sqrt{j(j+1)}$, etc.
b) lie on a core of height m,
spin _ $|S_1 m_1, S_2 m_2\rangle$: $|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\rangle = \alpha_1\alpha_2$, $|\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\text{-}\tfrac{1}{2}\rangle = \alpha_1\beta_2$, β_1, α_1, $\beta_1\beta_2$
$|S_1 m_2, Sm\rangle$: $|\tfrac{1}{2}\tfrac{1}{2}, 1, 1\rangle = \alpha_1\alpha_2$ $|\tfrac{1}{2}\tfrac{1}{2}, 1, 0\rangle = \tfrac{1}{\sqrt{2}}(\alpha_1\beta_2 + \alpha_2\beta_1)$
in phase _ triplet
triplet/single: $|\tfrac{1}{2}\tfrac{1}{2}, 1, -1\rangle = \beta_1\beta_2$ $|\tfrac{1}{2}\tfrac{1}{2}, 0, 0\rangle = \tfrac{1}{\sqrt{2}}(\alpha_1\beta_2 - \alpha_2\beta_1)$

So have established by inspection
$|j, j_2\ jm\rangle =\quad c_{m_1 m_2} |j_1, jm_1\ j_2 m_2\rangle$
For case $j_1 = j_2 =$ _
How do it formally:
Know $m_{max} = m_{1max} + m_{2max}$ -- one way
operate lowers op

$$M - |j_1, j_2, j_1 + j_2, j_1 + j_2\rangle = (M_- M_{2-}) |j_1\ j_1, j_2\ j_2\rangle$$

$$\{j(j+1) - m(m-1)\}^{1/2}\hbar|j, j-1\rangle = \{j_1(j_1+1) - m_1(m_1-1)\}^{1/2}\hbar|j, j_1 - 1, j_2, j_2\rangle$$
$$+\{j_2(j_2+1) - m_2(m_2-1)\}^{1/2}\hbar|j_1\ j_1; j_2, j_2 - 1\rangle$$

This gives you an expression for C_{m1m2}

ex: $S_-\alpha_1\beta_2 = S_-|11\rangle = \{2 + 1?0\}^{1/2}\hbar|10\rangle$
$(S_1 + S_2)_{\alpha_1\alpha_2} = \hbar\beta_1\alpha_2 = \hbar\alpha_1\beta_2$
$|1\ 0\rangle = \tfrac{1}{\sqrt{2}}(\beta_1\alpha_2 + \alpha_1\beta_2)$ as before
what about $|0\ 0\rangle$ _ must be orthogonal to $|10\rangle$
so $\tfrac{1}{\sqrt{2}}[\alpha_1\beta_1 - \alpha_1\beta_2] = |0\ 0\rangle$

More?

general concept: multiply equation above by $\langle j_1 m_1, j_2 m_2 |$

$\langle j_1 m_1, j_2 m_2 | j_1 m_2, j\, m \rangle = C_{m_1 m_2}$

Terms:

Coupling of orbital angular momenter for two electrons

$\ell_1 + \ell_2 = \ell_{max} = 0,\ 1,\ 2,\ 3,\ 4,\ 5 \qquad |1,1,2,M\rangle$ ♦

$\qquad\qquad\qquad\qquad S\ \ P\ \ D\ \ F\ \ G\ \ H \qquad |1_1, 1_2, D, M\rangle$

Multiple electrons:

Couple $j_1 + j_2 = j_{12}$ then all $j_{12} + j_3 = j_{12}$, etc.

$j_{12} = (j_1 + j_2), (j_1 + j_2 - 1), \ldots, |j_1 - j_2|;\ \ j_{12} = (j_{12} + j_3), \ldots, |j_{12} - j_3|$ for all