Quantum coding and decoding in multi-level adiabatic passage schemes

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Abstract

We present quantum schemes that can code/decode information by optical adiabatic passage processes. In a quantum decoder, information encoded on \( N \) levels (quNit), with population amplitudes of \( M \) possible phases, is processed by transferring their population optically to (predominantly) one of \( L \) final levels. In this two-photon adiabatic passage, \( N \) intermediate levels are resonantly coupled 'one-to-one' to \( N \) initial states and 'one-to-all' to \( L \) final states. A quantum encoder works in the opposite way. We discuss practical implementation of the suggested schemes in manifolds of vibrational states of the Na\(_2\) dimer.

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1. Introduction

Practical realization of many challenging problems in quantum computation [1], quantum communication [2] and use of quantum devices [3,4] is limited by decoherence effects [5]. Various approaches have been suggested to overcome this problem, like using of robust codes [6,7] or decay-free states [8]. It is thus of interest to search systems that can perform general unitary operations [9] and convert quantum information between different codes.

In this work, we present new schemes that can efficiently code and decode quantum information, when applied in molecular systems. They are based on robust two-photon adiabatic passage techniques [10–12], which are generalized to include many initial, intermediate and final states. This step allows selective addressing of many final states, according to a priori unknown initial superposition states. Analogous multi-level schemes can be used in preparation of vibrational wave packets in molecular dimers [13], in chiral purification of ‘racemic’ mixtures of enantiomers [14] or in control of isomerization of Jahn–Teller molecules [15].

2. Quantum decoder

In the quantum decoder, which is shown in Fig. 1, information is encoded on \( N \) initial energy levels, \( |\tilde{\psi}_i\rangle \) \((i=1,\ldots,N)\), with \( M \) distinct phases of their population amplitudes, that form ‘discrete’ quNit states [16]. The initial \( |\tilde{\psi}_i\rangle \) states are resonantly coupled one-to-one to \( N \) intermediate \( |\tilde{\psi}_n\rangle \) states, by a ‘pump’ pulse of \( \Omega_{AB}^{(n)} \) Rabi frequencies. These states are in turn resonantly coupled one-to-all to a (larger) set of final \( |\tilde{\phi}_k\rangle \) states \((k=1-L)\), by a ‘dump’ pulse of \( \Omega_{BC}^{(k)} \) Rabi frequencies.

In this two-photon multi-path adiabatic passage, we use a ‘counter-intuitive’ pulse ordering [10–12], in which the ‘dump’ (D) pulse precedes the ‘pump’ (P) pulse. Decoding is realized by passing the population on the \( N \) initial \( |\tilde{\psi}_i\rangle \) states to predominantly one of the \( L \) final \( |\tilde{\phi}_k\rangle \) states.
The coupling is mediated by multimode electric fields
\[ E_{P(D)}(t) = R_c \sum_{k,l} E_{k,l}^{AB}(BC) t) e^{-i\omega_{k,l}^{AB}(BC)t}. \] (1)

Thus, \( \Omega_{k,l}^{AB}(BC) (t) \equiv \mu_{k,l}^{AB}(BC) E_{k,l}^{AB}(BC) (t) \equiv \Omega_{k,l}^{AB}(BC) f_{P(D)}(t) \) are the Rabi frequencies, where we use the electric-dipole matrix-elements \( \mu_{k,l}^{AB}(BC) \) between the \( |k\rangle_{AB} \) and \( |\Omega_{k,l}^{BC} \rangle \) states, and \( 0 < f_{P(D)}(t) < 1 \) are the common envelopes of the dump and pump pulses.

The total Hamiltonian, in the rotating waves approximation and neglecting off-resonance terms, is \( (\hbar = 1) \)
\[
H = \sum_{l=1}^{2N+L} \omega_{l} \langle \ell | \ell \rangle + \sum_{l=1}^{N} \left[ \Omega_{l}^{AB}(t) e^{-i\omega_{l}^{AB}(t)} | \ell \rangle_{B} \langle \ell |_{A} + h.c. \right] + \sum_{l=1}^{N} \left[ \Omega_{l}^{BC}(t) e^{-i\omega_{l}^{BC}(t)} | k \rangle_{C} \langle l |_{B} + h.c. \right]. \] (2)

In the first sum, \( \ell \) goes over all states of the system, with energies \( \omega_{l} \). The second term represents the one-to-one resonant coupling of the initial states to the intermediate states, by the pump pulse whose frequency components are \( \omega_{l}^{AB} \equiv \omega_{l} - \omega_{A} \). The third term represents the one-to-all resonant coupling of the intermediate states to the final states, by the dump pulse with frequency components \( \omega_{l}^{BC} \equiv \omega_{C} - \omega_{l} \). We assume that the level energies are such that only the described couplings are resonant and parasitic effects due to non-resonant neighboring couplings can be neglected [13].

We expand the system wave function in the material states as, \( |\psi(t)\rangle = \sum_{k,l} \langle \psi | e^{-i\omega_{l}^{k,l} t} | k \rangle \). The column vector of slow-varying coefficients \( \mathbf{c} = (c_1, c_2, \ldots) \) is a solution of the matrix-Schrödinger equation \( \mathbf{c}(t) = -iH(t) \cdot \mathbf{c}(t) \). The effective time-dependent Hamiltonian is
\[
H(t) = \begin{bmatrix}
0 & H_{AB}^{\ell} & 0 \\
H_{AB}^{\ell} & 0 & H_{BC}^{\ell} \\
0 & H_{BC}^{\ell} & 0
\end{bmatrix},
\] (3)
where \( H_{AB}^{\ell} = \Omega_{l}^{AB} \delta_{\ell} \) is diagonal and \( H_{BC}^{\ell} = \Omega_{l}^{BC} \) a full matrix. We can find that \( H(t) \) has \( 2N \) eigenvalues, of which at most \( 2N \) are nonzero, \( \lambda_{\ell} \neq 0 \), and the rest \( L = 0 \), \( \lambda_{2N+1} \neq 0 \), so they correspond to ‘null’ states. Re-defining the basis as \( e^{-i\omega_{l}^{k,l} t} | \ell \rangle \)

\[
\text{we can check from } H \mathbf{d}_{k} = 0 \quad (k = 1 - L) \text{ that these states are characterized by the time-dependent vectors,}
\]
\[
\mathbf{d}_{1} = \left( \Omega_{11}^{AB} / \Omega_{11}^{AB}, \ldots, \Omega_{11}^{BC} / \Omega_{11}^{BC}, 0, \ldots, 0, -1, 0, \ldots, 0 \right),
\]
\[
\mathbf{d}_{2} = \left( \Omega_{12}^{AB} / \Omega_{11}^{AB}, \ldots, \Omega_{12}^{BC} / \Omega_{12}^{BC}, 0, \ldots, 0, 0, -1, \ldots, 0 \right),
\]
\[
\vdots
\]
\[
\mathbf{d}_{L} = \left( \Omega_{L1}^{AB} / \Omega_{11}^{AB}, \ldots, \Omega_{L1}^{BC} / \Omega_{11}^{BC}, 0, \ldots, 0, 0, 0, \ldots, -1 \right).
\] (4)

In order to simplify the analysis, we can assume, without loss of generality [13], that \( \Omega_{11}^{AB} = \Omega_{22}^{AB} = \ldots = \Omega_{NN}^{AB} = \Omega_{AB} \). Then, the first \( N \) coefficients in the \( \mathbf{d}_{k}(t) \) null states are solely determined by the dump vectors \( \mathbf{D}_{k} = \left( \Omega_{11}^{BC}, \ldots, \Omega_{NN}^{BC} \right) \).

The \( \mathbf{d}_{k}(t) \) states correlate one-to-one with the \( |1-N\rangle \) final states. Thus, if we manage to initially populate only one of the \( \mathbf{d}_{k}(t) \) states, adiabatic following would exclusively pass this population to a single final \( |k\rangle_{C} \) state, in the end of the process, so the decoding process would be perfect.

In reality, the initial population of the \( |1-N\rangle \) states splits among the \( (k=1-L) \) dark states, according to the squared projections, \( p_{k} = |c_{0}^{k} \rangle \Omega_{k}^{AB} | \mathbf{D}_{k} \rangle^{2} = \langle c_{0}^{k} \rangle \Omega_{k}^{AB} | \mathbf{D}_{k} \rangle^{2} \), of the vector of initial coefficients \( c_{0} = (c_{0}, c_{2}, \ldots, c_{N}) \) on the dump vectors \( \mathbf{D}_{k} \). Analogous relation determines the fraction of population transferred through a single intermediate level [13], by many pump Rabi frequencies. Therefore, we can maximize the population \( p_{k} \), transferred to the chosen (decoded) \( |k\rangle_{C} \) state, by using \( \Omega_{k}^{AB} = c_{0} \). At the same time, parasitic transfers to other \( |k\rangle_{C} \) \( (k \neq i) \) states could be minimized if the \( \Omega_{k}^{AB} \) and \( \Omega_{k}^{BC} \) vectors are orthogonal, \( \Omega_{k}^{AB} \cdot \Omega_{k}^{BC} \approx 0 \). The first condition is easy to meet, while the last cannot be fully realized if \( N < L \). This is because, we cannot orthogonalize all the \( L \) vectors, \( \Omega_{k}^{AB} \), in a \( N( < L) \) dimensional space. Only if \( N = L \), their components can be chosen such that the \( L \) dark states are orthogonal one to another, in the form given in Eq. (4), and the transfers to the final \( |k\rangle_{C} \) states are complete and exclusive. Very good coding/decoding machines can still be realized, unless \( N < L \).

As an example, we can examine the transfer of number information stored in \( M = 2 \) phases of the coefficients of \( N \) equally populated initial levels \( |i\rangle_{A} \). In this ‘binary-phase’ coding the binary number \( (0,0,\ldots,0) \), i.e., \( 0 \cdot 2^{0} + 0 \cdot 2^{1} + \ldots + 0 \cdot 2^{N} = 0 \), is represented by the amplitude \( c_{i} = (1,1,\ldots,1) \) of the \( |i\rangle_{A} \) states, the binary number...
(1,0,...,0), i.e., $1 \cdot 2^0 + 0 \cdot 2^1 + \cdots + 0 \cdot 2^N = 1$, by $c_2 = (-1,1,...,1)$, and so on. Because quantum states are known up to an overall phase, we can encode in this manner on $N$ levels only $2^{N-1}$ numbers.

As mentioned above, the decoding can be optimized by aligning Rabi vectors $\Omega_{D_k}$, which dump population to the $|k\rangle_C$ states, with the related $c_k$ vectors of encoded coefficients, $\Omega_{D_k} \approx c_k$. Accordingly, the $|1\rangle_C$ state is coupled to the intermediate $|1-N\rangle_B$ states by the vector of Rabi amplitudes $O_{D_1} = (1,1,...,1) = c_1$, the $|2\rangle_C$ state by $O_{D_2} = (-1,1,...,1) = c_2$, and so on.

In Fig. 2, we present the quantum decoder. The magnitudes of all the Rabi frequencies are $|O_{ij}| = 30\pi$, where $f_{DMP}(t) = \exp[-(t-t_{DMP})^2/r^2]$ and $t_p - t_0 = 2\tau$. The system decodes each of the binary-phase stored number by transferring the population of the initial superposition state, representing this number, to predominantly one of the (16) final $|k\rangle_C$ eigenstates. The degree of exclusivity of the transfer is such that 5 other final states end up with 2.78 times less population, while the remaining 10 states being considerably less populated.

We can obtain analytically the populations $p_k$ of the final $|k\rangle_C$ states, that are proportional to the squared scalar products, $p_k = |c_0 \cdot \Omega_{D_k}|^2$, of the encoded initial vector $c_0 = (c_1, c_2, ..., c_N)$ with the dump Rabi vectors $\Omega_{D_k}$. For example, by encoding the first 'hexagonal' level, $c_0 = (1,1,1,1,1)$, and taking into account that $O_{D1} = (1,1,1,1,1)$ and $O_{D2} = (-1,1,1,1,1)$, we obtain that $p_1 \propto |c_0 \cdot O_{D1}|^2 = 25$ and $p_2 \propto |c_0 \cdot O_{D2}|^2 = 9$, so that $p_1 / p_2 = 5/3 \approx 2.78$, as in Fig. 2. From the scalar products $c_0 \cdot O_{D1}, c_0 \cdot O_{D2}$, we can get all the probabilities, $p_{1-16}$: five of them are $\frac{5}{2} p_1$ and 10 are $\frac{1}{2} p_1$. Thus, from the normalization factor $n = 1 + 5(9/25) + 10(1/25) = 3.2$, we obtain $p_1 = n^{-1} = 0.3125$, determining the populations $p_k$ in all the cases.

Let us now generalize the decoding devices. We can increase the density of coding [17], by using the initial 'discrete' quNit states, with amplitudes of $M$ different phases of the $\exp(2\pi i/M)$ roots of the identity. Therefore, a system with $N$ initial levels and information stored in $M$ possible phases can be prepared in $M^{N-1}$ states and transferred to the same number of final levels.

In Tables 1 and 2, we present for selected systems the maximal population $p_{\text{max}}$ and the ratio $r = p_{\text{max}} / p_{\text{next}}$ of the maximal and next to maximal populations on the final states. We have calculated them as in Fig. 2, by numerically enumerating all the possible states, and confirming the results by dynamical simulations. The ratio $r$, which determines the decoding selectivity, can be considered to be good if $r > 2$, that is for systems above or on the diagonal in Tables 1 and 2, given by $N = 6$ and $M = 4$. Clearly the decoding is better the closer the number of the $N$ initial/intermediate and $M^{N-1}$ final states is, i.e. the smaller $N$ and $M$ are. The efficiency of en/decoding in these systems could approach that in standard quantum gates [1], if the pulses are tailored to the processed quantum information.

Since relaxation is neglected, the (unitary) evolution is invertible. Thus the population can be transferred back by the same set of pulses using a time-reversed dump–pump pulse sequence. In practice, we could detect the final decoded state by monitoring its fluorescence to some lower lying state, but after the measurement the coherence is largely lost. Of a large practical interest is the question, whether we can use

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**Table 1**

<table>
<thead>
<tr>
<th>$N$</th>
<th>$p_{\text{max}}$</th>
<th>$p_{\text{max}}/p_{\text{next}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.666</td>
<td>0.26</td>
</tr>
<tr>
<td>3</td>
<td>0.250</td>
<td>0.14</td>
</tr>
<tr>
<td>4</td>
<td>0.144</td>
<td>0.09</td>
</tr>
<tr>
<td>5</td>
<td>0.066</td>
<td>0.06</td>
</tr>
<tr>
<td>6</td>
<td>0.025</td>
<td>0.02</td>
</tr>
</tbody>
</table>

Here, $N$ is the number of levels and $M$ the number of phases used.

**Table 2**

<table>
<thead>
<tr>
<th>$r$</th>
<th>$N=2$</th>
<th>$N=3$</th>
<th>$N=4$</th>
<th>$N=5$</th>
<th>$N=6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M=2$</td>
<td>$\infty$</td>
<td>9</td>
<td>4</td>
<td>2.78</td>
<td>2.25</td>
</tr>
<tr>
<td>$M=3$</td>
<td>4</td>
<td>3</td>
<td>2.29</td>
<td>1.92</td>
<td>1.71</td>
</tr>
<tr>
<td>$M=4$</td>
<td>2</td>
<td>1.8</td>
<td>1.6</td>
<td>1.47</td>
<td>1.38</td>
</tr>
</tbody>
</table>
the inverted pulse sequence for encoding numbers into the phases, by starting out just with a single populated ‘hexadecimal’ level.

3. Quantum encoder

In Fig. 3, we display the results of this quantum encoding. In each run, we initially populate one of the (16) ‘hexadecimal’ $|k\rangle_C$ levels, shown in Fig. 2, and evolve the system by the time reversed pulse sequence. In the front of Fig. 3, we see the binary-phase coded amplitudes of the (5) $|\alpha\rangle_A$ levels. In the back, we show populations left on the ‘hexadecimal’ levels. In contrast to the quantum converter scheme, the coding here is perfect, since we obtain the correct phases of the $|\alpha\rangle_A$ states.

However the transfer is far from complete, since about 50% of the population is transferred: 31% are transferred to the initially populated level as, $\langle 1 | - \rangle = (1,1,1,1,1)$, instead of $\langle 1 | - \rangle = (1,1,1,1,1)$. Then, $c_9$ is linearly dependent on the other three vectors, $c_4 = c_9 + c_3 - c_1$, and the reflected populations are $p_4 = 0.5$, while the population of the $|A\rangle$ state, $p_A = 0.5$, correlated to the INS, remains intact.

This system can be extended in a chain-like manner to the case in which levels $|1\rangle$, $|2\rangle$, $|3\rangle$ are coupled to levels $|4\rangle$, $|5\rangle$ by the pump pulses $\Omega_{14} = \Omega_{24} = \Omega_{25} = \Omega_{35}$, with levels $|4\rangle$ and $|5\rangle$ coupled to $|6\rangle$ and $|7\rangle$ by the dump pulse $\Omega_{46} = \Omega_{57}$. If any of the $|1\rangle - |3\rangle$ states is initially populated, 2/3 of the population is transferred and 1/3 remains intact. This can be seen by resolving the initially populated level as, $|2\rangle = (|2\rangle - |1\rangle - |3\rangle) + |1\rangle + |3\rangle$. From Eq. (4), we find that one MNS correlates level $|1\rangle$ to $|6\rangle$, the other correlates level $|3\rangle$ to $|7\rangle$, while the INS is always correlated to $|2\rangle - |1\rangle - |3\rangle$. By following this chain of ‘frustrated’ transfers, we obtain that the total population transferred in each case is $1/2, 2/3, 3/4, ...$, i.e., the ratio of the number of intermediate levels and the number of initial levels. This ‘quantum reflection law’ is rather general, as we show below.

We can examine analogously the MNS and INS states in Fig. 3. We initially consider only the $|1\rangle - |4\rangle_C$ states, with $|1\rangle_C$ initially populated. Their coupling vectors are $c_1 = (1,1,1,1,1)$, $c_2 = (-1,1,1,1,1)$ and $c_4 = (1,1,1,1,1)$ and $c_6 = (-1,1,1,1,1)$. Then, $c_4$ is linearly dependent on the other three vectors, $c_4 = c_9 + c_3 - c_1$, and the reflected populations are $p_4 = 1/16$. If we wish to encode the vector $c_1 = (1,1,1,1,1)$, instead of $c_4$, the $c_2,3,5$ vectors would be linearly independent on $c_1$. Therefore, no INS correlating to the $|1\rangle_C$ state can be formed, and the population is fully transferred. When we also add
the $|8\rangle_C$ state, with $c_8 = (-1, -1, -1, -1, 1) = c_2 + c_3 + c_5 - 2c_1$, the INS correlates more with the $|1\rangle_C$ state and the reflected populations are $p_2 = p_3 = p_5 = p_8 = p_1/4$. By adding more and more states, we finally match Fig. 3, where all these cases interfere and give a large reflected population, mostly left on the $|1\rangle_C$ state.

We can find the probability $P_{\text{tran}}$ of transfer from a single (decoded) level to all the (encoded) levels in general systems. Then, among the $M^{N-1}$ null states, $N$ are MNS and $M^{N-1} - N$ are INS. In accordance with the above quantum reflection law, we might expect that $P_{\text{tran}} = N/M^{N-1}$. Dynamical simulations fully confirm this fact. Surprisingly, these values are identical with those in Table 1, i.e. $P_{\text{tran}} = p_{\text{max}}$. The same holds for the probability of transfer to the ‘incorrectly decoded’ levels and back; in the last case, we need to project the transferred population amplitudes on the initially encoded amplitudes. These adiabatically driven systems thus behave as ‘passive’ multi-channel elements, with the same transfer probabilities in both directions.

4. Practical realization

We can apply the above de/coding schemes in manifolds of molecular vibrational states, such as those found in the Na$_2$ dimer [13]. The initial vibrational wave packet, $|P(0)\rangle = \sum_{i=1}^{M} c_i^0 |X^1\Sigma^+_g, v\rangle$, is sitting on the ground electronic state $X^1\Sigma^+_g$ of the Na$_2$ molecule [18], where $c_i^0 = \pm 1$ are chosen according to the encoded number. It could be prepared in simpler multi-level adiabatic schemes [13]. The $|X^1\Sigma^+_g, v = 1 - M\rangle$ vibrational states are coupled one-to-one to the intermediate $|A^1\Sigma^+ u, v = 1 - M\rangle$ vibrational states, sitting on the excited electronic state $X^1\Sigma^+$. These are then coupled one-to-all to the $|X^1\Sigma^+_g, v = (M + 1) - (M + L)\rangle$ vibrational states. Since the vibrations are not harmonic, the resonant transitions should not interfere one with another, if the pulses are long enough [13].

The resonant electric field components $E_{v',v}^{\text{res}}(t)e^{-i\omega_{v',v}t}$, introduced in Eq. (1), have the amplitudes $E_{v',v}^{\text{res}}(t) = f_D(t)\delta_{v',v} C/\mu_{v',v} (\delta_{v',v} = 1, 0$ if $v' = v$ resp. $v' \neq v$) and $E_{v,v}^{\text{res}}(t) = f_p(t) C \mu_{v',v} / \mu_{v,v}$, where the transition-dipole matrix elements are [19,20] $\mu_{v',v} = \langle A^1\Sigma^+_u | v' | X^1\Sigma^+_g, v \rangle$. For $c_{v',v}^0 = \pm 1$, chosen according to the decoding structure, the adiabaticity can be satisfied for $C = 30/\pi \equiv 1$ ps$^{-1}$ and $f_D(t)$ as in Fig. 2. With this choice, we obtain the above Rabi frequencies, since $\Omega_{v',v}^{\text{res}} = \mu_{v',v} E_{v',v}^{\text{res}}(t) = f_D(t) C (v' = v)$ and $\Omega_{v,v}^{\text{res}} = \mu_{v,v} E_{v,v}^{\text{res}}(t) = f_p(t) C \mu_{v',v} / \mu_{v,v}$. This shows that small molecules, such as the Na$_2$ dimer, could be used to realize the suggested de/coding schemes.

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References


